Hermitian operators

An operator $\hat{A}$ having two eigenfunctions ψ_1 and ψ_2 is said to be Hermitian if

$$\int \psi_1 (\hat{A}\psi_2)\, d\tau = \int \psi_2 (\hat{A}\psi_1)\, d\tau$$

When ψ_1 and ψ_2 are real.

Or,
$$\int \psi_1^* (\hat{A}\psi_2)\, d\tau = \int \psi_2 (\hat{A}\psi_1)^* d\tau$$

When ψ_1 and ψ_2 are complex. Here ψ_1^* is the complex conjugate of ψ_1 and $d\tau$ is the volume element of space in which the functions ψ_1 and ψ_2 defined.

The functions e^{ix} (ψ_1) and $\sin x$ (ψ_2) are the two acceptable eigenfunction of the operator $\frac{d^2}{dx^2}$ ($\hat{A}$).

Now,
$$\int \psi_1^* (\hat{A}\psi_2)\, d\tau = \int e^{-ix}\left(\frac{d^2}{dx^2}\sin x\right) dx$$
$$= -\int e^{-ix} \sin x \, dx \quad \text{--- (1)}$$

and,
$$\int \psi_2 (A\psi_1)^* d\tau = \int \sin x \cdot \left[\frac{d^2 (e^{ix})^*}{dx^2}\right] dx$$
$$= \int \sin x \, (i^2 e^{ix})^* dx$$

$$= -\int \sin x \, e^{-ix} \, dx \quad \text{——— (2)}$$

from eq^n (1) and (2)

$$\int \psi_1^* (\hat{A} \psi_2) d\tau = \int \psi_2 (A \psi_1)^* d\tau$$

Hence, the operator $\hat{A} = \frac{d^2}{dx^2}$ is a Hermitian operator.

* The eigen values of a Hermitian operators are real. $\Rightarrow$

Let, $\hat{A}$ be a Hermitian operator with eigen function ψ and corresponding eigen value a.

Now, we have

$$\hat{A} \psi = a \psi \quad \text{——— (1)}$$

and

$$(\hat{A} \psi)^* = a^* \psi^* \quad \text{——— (2)}$$

Multiplying the eq^n (1) with ψ^* and integrating, we have

$$\int \psi^* \hat{A} \psi d\tau = \int \psi^* a \psi d\tau$$

$$= a \int \psi^* \psi \, d\tau \quad \text{——— (3)}$$

Again multiplying the eqn. 2 with ψ and then integrating, we get

$$\int \psi (\hat{A}\psi)^* d\tau = \int \psi a^* \psi^* d\tau$$

$$= a^* \int \psi^* \psi \, d\tau \quad \text{——} (4)$$

Since $\hat{A}$ is Hermitian operator,

$$\int \psi^* \hat{A} \psi \, d\tau = \int \psi (\hat{A}\psi)^* d\tau \quad \text{——} (5)$$

from eqn. 3, 4 and 5, we have

$$a \int \psi \psi^* d\tau = a^* \int \psi \psi^* d\tau$$

or,

$$a = a^*$$

It is possible only when a is real.

Thus, a Hermitian operator has real eigen values.

Numericals based on Hermitian operator

1.⇒ Examine if d^2/dx^2 is a Hermitian operator.

Solⁿ ⇒ for an operator to satisfy the Hermitian condition we say that if an operator $\hat{A}$ has two eigenfunctions ψ and ϕ and if

$$\int \psi(\hat{A}\phi)d\tau = \int (\hat{A}\psi)\cdot\phi \, d\tau \quad (\psi \text{ and } \phi \text{ are real})$$

or,
$$\int \psi^*(\hat{A}\phi)d\tau = \int (\hat{A}\psi)^*\cdot\phi \, d\tau \quad (\psi \text{ and } \phi \text{ are Complex})$$

Then $\hat{A}$ is called the Hermitian operator.

Let,
$\psi = e^{ix}$ and $\phi = \sin x$ be the two acceptable eigenfunctions. Then,

$$\int \psi(\hat{A}\phi)d\tau = \int e^{-ix}\frac{d}{dx^2}(\sin x)\,dx = -\int e^{-ix}\sin x \, dx$$

$$\int \phi(\hat{A}\psi)^* d\tau = \int \sin x\left[\frac{d^2}{dx^2}(e^{ix})\right]^* dx = \int \sin x\,(i^2 e^{ix})^* dx$$

$$= -\int \sin x \, e^{-ix}\, dx$$

Since the two integrals are the same, $\frac{d^2}{dx^2}$ is Hermitian.

2.⇒ Show that the operator $\hat{P}_x$ for Linear momentum is Hermitian.

Solⁿ ⇒ It is required to prove that

$$\int_{-\infty}^{\infty} \psi^*\left(-i\hbar\frac{d}{dx}\right)\phi \, dx = \int_{-\infty}^{\infty} \phi\left(-i\hbar\frac{d}{dx}\right)^* \psi^* \, dx$$

Integrating by parts, the left hand side is equal to

$$-i\hbar\left[\psi^*\phi\right]_{-\infty}^{\infty} - (i\hbar)\int_{-\infty}^{\infty}\phi \frac{d\psi^*}{dx}\,dx = 0 + \int_{-\infty}^{\infty}\phi\left(i\hbar\frac{d}{dx}\right)\psi^* dx$$

$$= \int_{-\infty}^{\infty}\phi\left(-i\hbar\frac{d}{dx}\right)^*\psi^* dx = R.H.S.$$

Hence $\hat{p}_x$ is a Hermitian operator.

3. $\Rightarrow$ Show that if two operators $\hat{A}$ and $\hat{B}$ are Hermitian, then their product $(\hat{A}\hat{B})$ is also Hermitian if and only if $\hat{A}$ and $\hat{B}$ commute.

Sol^n $\Rightarrow$

Since $\hat{A}\hat{B} = \hat{B}\hat{A}$ (given)

$$\int\psi^*(\hat{A}\hat{B})\phi\,d\tau = \int\psi^*(\hat{B}\hat{A})\phi\,d\tau = \int\psi^*\hat{B}(\hat{A}\phi)\,d\tau$$

Since $\hat{B}$ is Hermitian, the integral is equal to

$$\int(\hat{A}\phi)(\hat{B}\psi)^*d\tau = \int(\hat{B}\psi)^*(\hat{A}\phi)\,d\tau = \int\phi\hat{A}^*(B\psi)^*\,d\tau$$

$$= \int\phi(\hat{A}\hat{B})^*\psi^*\,d\tau \quad (\because \hat{A} \text{ is also Hermitian})$$

Hence,

$\hat{A}\hat{B}$ is also Hermitian.

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